1. Boolean Algebra

Definition: Let B be non-empty set with two binary operations + & · , one unary operation (complement), two distinct elements 0 and 1.

$(B, +, \cdot, ', 0, 1)$

0 - Least Element
1 - Greatest Element.

& B is said to Boolean Algebra if it satisfied following axioms or laws,

$+ = \vee$
$\cdot = \wedge$

i) Closure Law :- $\forall\ a, b \in B$

$$a + b \in B$$
$$a \cdot b \in B$$

ii) Associative Law :- Let $a, b, c \in B$

$$(a+b)+c = a+(b+c)$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

iii) Identity Law :- $\forall\ a \in B$

$$a + 0 = a$$
$$a \cdot 1 = a$$

iv) Complement Law :- $\forall\ a \in B$

$$a + \bar{a} = 1$$
$$a \cdot \bar{a} = 0$$

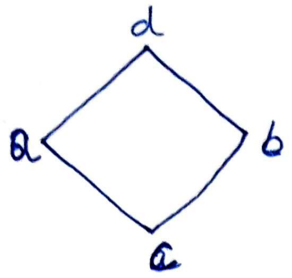
v) Distributive Law :- $a + (b \cdot c) = (a+b) \cdot (a+c)$
$\forall\ a, b, c \in B$ $\quad a \cdot (b+c) = a \cdot b + a \cdot c$

vi) Commutative Law :- $\forall\ a, b \in B$

$$a + b = b + a$$

$$ab = ba$$

<u>example</u> :-
1.



$B = \{a, b, c, d\}$

<u>Sol</u>: least Element = c = 0

Greatest Element = d = 1

i) Closure Law :- Let $a, b \in B$

$a + b$ i.e. $a \vee b = d \in B$

$a \cdot b$ i.e. $a \wedge b = c \in B$

closure Law holds.

ii) Associative Law :- Let $a, b, c \in B$

L.H.S. $(a+b)+c = d + c = d$

R.H.S. $a + (b+c) = a + b = d$

L.H.S = R.H.S

L.H.S. $a \cdot (b \cdot c) = a \cdot c = c$

R.H.S. $(a \cdot b) \cdot c = c \cdot c = c$

L.H.S = R.H.S.

Associative Law holds.

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iii) Identity Law :: $a+0$ ie. $a \vee c = a$

$a \cdot 1$ ie. $a \wedge d = a$

$\therefore$ I.L. Holds

iv) Complemented Law :- $a + \bar{a}$ ie. $a \vee b = d = 1$

$a + \bar{a} = 1$

$\left[\because \begin{matrix} a \vee b = d = 1 \\ a \wedge b = c = 0 \end{matrix} \quad \therefore \begin{matrix} a^c = b \\ b^c = a \end{matrix} \right]$

$a \cdot \bar{a}$ i.e. $a \wedge b = c = 0$

$\therefore \quad a \cdot \bar{a} = 0$

C.L. holds.

v) Commutative Law :- Let $a, b \in B$

$a+b$ ie. $a \vee b = d$

$b+a$ ie. $b \vee a = d$

$\therefore \quad a+b = b+a$

IIly $a \cdot b = c$

$b \cdot a = c$

$\therefore \quad a \cdot b = b \cdot a$

vi) Distributive Law :- Let $a, b, c \in B$

L.H.S $a + (b \cdot c) = a + c = a$

R.H.S. $(a+b) \cdot (a+c) = d \cdot a = a$

IIly $a \cdot (b+c) = a \cdot b = c$

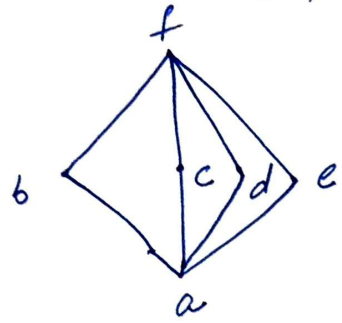
$a \cdot b + a \cdot c = c + c = c$

$\therefore$ D.L. holds.

Hence Given B is Boolean Algebra.

2. Boolean Algebra as Lattice :- A Lattice is said to be Boolean Algebra if it is both distributive and Complemented. Complement of Each Element is unique.

example 1.



least Element = a
Greatest Element = f

Sol/ Here $a \wedge f = a$ $\qquad \therefore a^c = f$

$a \vee f = f$ $\qquad f^c = a$

Also $b \vee d = f$ $\qquad \therefore b^c = d$

$b \wedge d = a$ $\qquad d^c = b$

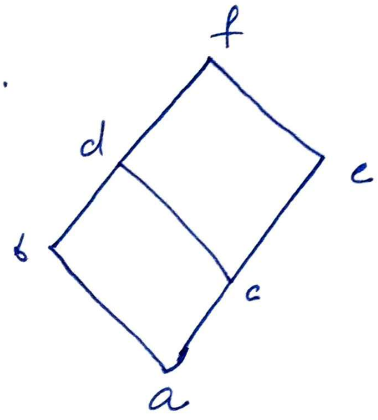
Now $b \vee c = f$ $\qquad \therefore b^c = c$

$b \wedge c = a$ $\qquad c^c = b$

b has not unique Complement

$\therefore$ It is not boolean algebra.

eg. 2.



Sol :- Least Ele. = a
Great. Ele. = f

$b \vee c = d$

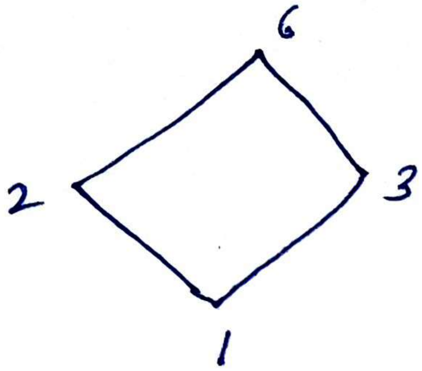
$b \wedge c = a$

$b \vee e = f$ $\qquad \therefore b^c = e$

$b \wedge e = a$ $\qquad e^c = b$

c has no Complement ∴ It is not Boolean Alg.

g. 3.



L.E = 1 , G.E = 6

$2 \vee 3 = 6$ ∴ $2^c = 3$

$2 \wedge 3 = 1$ $3^c = 2$

$1 \vee 6 = 6$ $1^c = 6$

$1 \wedge 6 = 1$ $6^c = 1$

∴ It is Boolean Algebra.